1. Roads A and B are the only escape routes from a state prison. Prison records show that, of the prisoners who tried to escape, 40% used road A, and 60% used road B. These records also show that 80% of those who tried to escape via A, 70% of those who tried to escape via B were captured.

   (a) What is the chance that one prisoner can successfully escape from the prison?

   (b) Suppose that two prisoners have independently and successfully escaped from the prison. What is the probability that they have used the same road to escape?

   Solution: (a) Chance, that the person captured via gate A is $= 0 \cdot 8 \times 0 \cdot 4 = 0 \cdot 32$. Chance, that the person captured via gate B is $= 0 \cdot 7 \times 0 \cdot 6 = 0 \cdot 42$. Chance that the person can successfully escape $= (0 \cdot 4 - 0 \cdot 32) + (0 \cdot 6 - 0 \cdot 42) = 0 \cdot 26$.

   (b) Chance that 1 can escape via gate A is $= 0 \cdot 4 - 0 \cdot 32 = 0 \cdot 08$. Chance that 1 can escape via gate B is $= 0 \cdot 6 - 0 \cdot 42 = 0 \cdot 18$. Therefore, the probability that the two prisoners have independently and successfully escaped from the prison by using the same road is $= \frac{(0 \cdot 08)^2}{(0 \cdot 26)^2} + \frac{(0 \cdot 18)^2}{(0 \cdot 26)^2} = \frac{388}{676}$.

   $\square$

2. Let N be the number of empty poles when r flags of different colours are displayed randomly on n poles arranged in a row (here $r, n \in \mathbb{N}$). Assuming that there is no limitation on the number of flags on each pole, compute the expectation of N.

Solution: Let us denote, $N_{n,r}$ be the number of empty poles when r flags of different colours are displayed randomly on n poles arranged in a row ($r, n \in \mathbb{N}$). We estimate $\mathbb{E}[N_{n,r+1}]$ in terms of $\mathbb{E}[N_{n,r}]$ by keeping an eye on $(r+1)^{th}$ flag. Let k denotes the number of empty slots after the first r flags. Note that, k can not be n as $r \geq 1$ and also $k = 0$ doesn't contribute to $\mathbb{E}[N_{n,r+1}]$. Therefore,

$$\begin{aligned}
&\mathbb{E}[N_{n,r+1}] \\
&= \sum_{k=1}^{n-1} kP[N_{n,r} = k]\left(1 - \frac{k}{n}\right) + (k-1)P[N_{n,r} = k]\frac{k}{n} \\
&= \sum_{k=1}^{n-1} \frac{k}{n} P[N_{n,r} = k](n - k + k - 1) \\
&= \frac{n-1}{n} \sum_{k=1}^{n-1} kP[N_{n,r} = k] \\
&= \frac{n-1}{n} \mathbb{E}[N_{n,r}].
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbb{E}[N_{n,r}] &= \left(\frac{n-1}{n}\right)^{r-1} \mathbb{E}[N_{n,1}] \\
&= \left(\frac{n-1}{n}\right)^{r-1} (n-1).
\end{aligned}$$

Here, we have used that $\mathbb{E}[N_{n,1}] = (n-1)$. Because, for 1 flag, with probability 1 there are $(n-1)$ empty poles.

$\square$

3. Suppose $X \sim Bin(n,p)$. Compute the 3^{rd} moment of X.

Solution:

$$\begin{aligned}
M(t) &= \mathbb{E}[e^{tx}] \\
&= \sum_{x=0}^{n} e^{tx} f(x) \\
&= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x} \\
&= \left[pe^t + (1-p)\right]^n .
\end{aligned}$$

$$M'(t) = n\left[pe^t + (1-p)\right]^{n-1} pe^t.$$

$$M''(t) = n\left[pe^t + (1-p)\right]^{n-1} pe^t + n(n-1)\left[pe^t + (1-p)\right]^{n-2} (pe^t)^2.$$

$$\begin{aligned}
M'''(t) &= n\left[pe^t + (1-p)\right]^{n-1} pe^t + n(n-1)\left[pe^t + (1-p)\right]^{n-2} (pe^t)^2 \\
&\quad + n(n-1)\left[pe^t + (1-p)\right]^{n-2} 2(pe^t)^2 + n(n-1)(n-2)\left[pe^t + (1-p)\right]^{n-3} (pe^t)^3.
\end{aligned}$$

Therefore,

$$M'''(0) = np + n(n-1)3p^2 + n(n-1)(n-2)p^3.$$

$\square$